EURASIAN BULLETIN OF MATHEMATICS
EBM (2018), VOL. 1, NO. 3, 107-116

Generalized η-Ricci solitons in Trans Sasakian manifolds

Mohd Danish Siddiqi

Department of Mathematics, Faculty of science, Jazan University, Jazan, Kingdom of Saudi Arabia

Abstract. The object of the present research is to study generalized η-Ricci soliton on trans-Sasakian manifolds. It shows that a trans-Sasakian manifold admitting the generalized gradient Ricci soliton, is necessarily Einstein manifold.
2010 Mathematics Subject Classifications: 53C15; 53C20; 53C25; 53C44.
Key Words and Phrases: Generalized η-Ricci Soliton, η-Ricci Soliton, Ricci soliton, Trans Sasakian manifold, Einstein manifold.

1. Introduction

Theory of Ricci solitons are the fixed points of the Ricci flow projected from the space of metrics onto its quotient modulo diffeomorphisms and scaling, and often arise as blow-up limits for the Ricci flow on compact manifolds. Theoretical physicists have also been looking into the equation of Ricci soliton in relation with string theory. The initial contribution in this direction is due to D. Friedan who discusses some aspects of it in [11].

In 1982, R. S. Hamilton [12] introduced that the Ricci solitons move under the Ricci flow simply by diffeomorphisms of the initial metric that is they are stationary points of the Ricci flow is given by

$$\frac{\partial g}{\partial t} = -2Ric(g). \tag{1.1}$$

Definition 1.1. *A Ricci soliton (g, V, λ) on a Riemannian manifold is defined by*

$$L_V g + 2S + 2\lambda g = 0, \tag{1.2}$$

where S is the Ricci tensor, L_V is the Lie derivative along the vector field V on M and λ is a real scalar. Ricci soliton is said to be shrinking, steady or expanding according as $\lambda < 0, \lambda = 0$ and $\lambda > 0$, respectively.

If the vector field V is the gradient of a potential function -ψ , then g is called a gradient Ricci soliton and equation (1.2) assumes the form $Hess\psi = S + \lambda g$.

On the other hand, the roots of contact geometry lie in differential equations as in 1872 Sophus Lie introduced the notion of contact transformation as a geometric tool to study systems of differential equations. This subject has manifold connections with the other fields of pure mathematics, and substantial applications in applied areas such as mechanics, optics, phase space of dynamical system, thermodynamics and control theory. The important of Ricci soliton comes from the fact

Email address: anallintegral@gmail.com , msiddiqi@jazanu.edu.sa

that they are corresponding to self-similar solutions of the Ricci flow and at the same time they are natural generalizations of Einstein metrics. Some generalizations like gradient Ricci solitons [7], quasi Einstein manifolds [8] and generalized quasi Einstein manifolds [9] play an important role in solutions of geometric flows and described the local structure of certain manifolds. In 2016, P. Nurowski and M. Randall [20] introduced the notion of generalized Ricci soliton as a class of overdetermined system of equations

$$\mathcal{L}_X g + 2a_1 X^\sharp \otimes X^\sharp - 2a_2 S - 2\lambda g = 0 \tag{1.3}$$

where $\mathcal{L}_X g$ and $X^\sharp$ denotes the Lie derivative of the metric g in the directions of vector field X and the canonical 1-form associated to X, and some real constants a_1, a_2 and λ .
It is a well known fact that, if the potential vector field ψ is zero or Killing then the Ricci soliton is an Einstein real hypersurfaces on non-flat complex space forms [6]. Motivated by this concept in 2009, J.T. Cho and M. Kimura [4] introduced the notion of η-Ricci solitons and gave a classification of real hypersurfaces in non-flat complex space forms admitting η-Ricci solitons.

Definition 1.2. *An η-Ricci soliton (g, V, λ, μ) on a Riemannian manifold is defined by*

$$\mathcal{L}_X g + 2S + 2\lambda g + 2\mu\eta \otimes \eta = 0, \tag{1.4}$$

where S is the Ricci tensor, $\mathcal{L}_X$ is the Lie derivative along the vector field X on M and λ is a real scalar. In particular $\mu = 0$ then the data (g, ξ, λ) is a Ricci soliton and it is said to be shrinking, steady or expanding according as $\lambda < 0, \lambda = 0$ and $\lambda > 0$, respectively.

Now, in the present paper, author is introducing the notion of generalized η-Ricci solitons. Perturbing the equation (1.3) that define, this type of soliton by the multiple of a ceratin $(0, 2)$-tensor field $\eta \otimes \eta$, we obtain more general notion, namely generalized η-Ricci soliton. Therefore, from the equations (1.3) and (1.4) we can extensively study the generalized Ricci solitons. The generalized η-Ricci soliton is given by the following equation, which is the association of above equations (1.2), (1.3) and (1.4),

$$\mathcal{L}_X g + 2a_1 X^\sharp \otimes X^\sharp - 2a_2 S - 2\lambda g - 2\mu\eta \otimes \eta = 0 \tag{1.5}$$

where $\mathcal{L}_X g$ and $X^\sharp$ denotes the Lie derivative of the metric g in the directions of vector field X and the canonical 1-form associated to X, and some real constants a_1, a_2 ,λ and μ.

In 1985, J. A. Oubina [22] introduced a new class of almost contact metric manifolds known as trans-Sasakian manifolds. This class consists of both Sasakian and Kenmotsu structures. The above manifolds are studied by several authors like D. E. Blair and J.C. Marrero ([1], [18]) K. Kenmotsu [16]. In 1925, Levy [17] obtained the necessary and sufficient conditions for the existence of such tensors. later, R. Sharma [24] initiated the study of Ricci solitons in contact Riemannian geometry . Followed by M. M. Tripathi [28], G. Nagaraja et.al. [21] and M. C. Turan and others [27] extensively studied Ricci solitons in almost contact metric manifolds.

η-Ricci soliton is the generalization of Ricci solitons. In [2] A. M. Blaga studied η-Ricci solitons on para-Kenmotsu manifolds and η-Ricci solitons on Lorentzian para Sasakian manifolds [3]. Later, D. G. Parakasha and B. S. Hadimani studied η-Ricci solitons on para-Sasakian manifolds [23]. In 2017, K. Venu and G. Nagaraja [30] study the η-Ricci solitons in trans Saskian manifolds. Also in 2017 M. A. Mekki and A. M. Cherif study the generalized Ricci solitons on Sasakian manifolds [19]. Recently, M. D. Siddiqi (see, [25], [26]) also, study some properties of solitons space which is closely related to this topic. Therefore, motivated by these studies in the present paper author introduces the notion of generalized η-Ricci soliton in trans-Sasakian manifolds. Trans-Sasakian manifolds arose in a natural way from the classification of almost contact metric structures by Chinae and Gonzales [10], and they appear as a natural generalization of both Sasakian and Kenmotsu manifolds.

2. Preliminaries

Let a differentiable manifold M is said to be an almost contact metric manifold equipped with almost contact metric structure (ϕ, ξ, η, g) consisting of a $(1,1)$ tensor field ϕ, a vector field ξ, a 1-form η and an indefinite metric g such that

$$\phi^2 X = -X + \eta(X)\xi, \;\; \eta(\xi) = 1, \;\; \eta \circ \phi = 0, \;\; \phi\xi = 0, \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \;\; \eta(X) = g(X, \xi). \tag{2.2}$$

for all $X, Y \in M$, $\xi \in \Gamma(TM)$ and 1-form $\eta \in \Gamma(\bar{T}M)$.

In [29], Tano classified the connected almost contact metric manifold. For such a manifold the sectional curvature of the plane section containing ξ is constant, say c. He showed that they can be divided into three classes. (1) homogeneous normal contact Riemannian manifolds with $c > 0$. Other two classes can be seen in Tano [29].

In [13], Grey and Harvella was introduced the classification of almost Hermitian manifolds, there appears a class W_4 of Hermitian manifolds which are closely related to the conformal Kaehler manifolds. The class $C_6 \oplus C_5$ [13] coincides with the class of trans-Sasakian structure of type (α, β). In fact, the local nature of the two sub classes, namely C_6 and C_5 of trans-Sasakian structures are characterized completely.

An almost contact metric structure on M is called a trans-Sasakian (*see* [13], [16]) if $(M \times R, J, G)$ belongs to the class W_4, where J is the almost complex structure on $M \times R$ defined by

$$J\left(X, f\frac{d}{dt}\right) = \left(\phi(X) - f\xi, \eta(X)\frac{d}{dt}\right)$$

for all vector fields X on M and smooth functions f on $M \times R$ and G is the product metric on $M \times R$. This may be expressed by the condition

$$(\nabla_X \phi)Y = \alpha(g(X, Y)\xi - \eta(Y)X) + \beta(g(\phi X, Y)\xi - \eta(Y)\phi X) \tag{2.3}$$

where α and β are some scalar functions. We note that trans Sasakian structure of type $(0,0)$, $(\alpha, 0)$ and $(0, \beta)$ are the cosymplectic, α-Sasakian and β-Kenmotsu manifold respectively. In particular, if $\alpha = 1, \beta = 0$ and $\alpha = 0, \beta = 1$, then a trans-Sasakian manifold reduces to a Sasakian and Kenmotsu manifolds respectively. From (2.3), it follows that

$$\nabla_X \xi = -\alpha\phi X + \beta[X - \eta(X)\xi], \tag{2.4}$$

and

$$(\nabla_X \eta)Y = -\alpha g(\phi X, Y) + \beta[g(X, Y) - \eta(X)\eta(Y)]. \tag{2.5}$$

for any vector fields X and Y on M, ∇ denotes the Levi-Civita connection with respect to g, α and β are smooth functions on M. The existence of condition (2.3) is ensured by the above discussion.

The Riemannian curvature tensor R with respect to Levi-Civita connection ∇ and the Ricci tensor S of a Riemannian manifold M are defined by:

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \tag{2.6}$$

$$S(X,Y) = \sum_{i=1}^{2n+1} g(R(X,e_i)e_i, Y) \tag{2.7}$$

for $X, Y, Z \in \Gamma(TM)$, where ∇ is with respect to the Riemannian metric g and $\{e_1, e_2, ...e_i\}$, where $1 \leq i \leq 2n+1$ is the orthonormal frame.

Given a smooth function ψ on M, the gradient of ψ is defined by:

$$g(grad\psi, X) = X(\psi), \tag{2.8}$$

and the *Hessian* of ψ is defined by:

$$(Hess\psi)(X,Y) = g(\nabla_X grad\psi, Y), \tag{2.9}$$

where $X, Y \in \Gamma(TM)$. For $X \in \Gamma(TM)$, we defined $X^\sharp \in \Gamma(\bar{T}M)$ by:

$$X^\sharp(Y) = g(X,Y). \tag{2.10}$$

The generalized η-Ricci soliton equation in Riemannian manifold M is defined by:

$$\mathcal{L}_X g = -2a_1 X^\sharp \odot X^\sharp + 2a_2 S + 2\lambda g + 2\mu\eta \otimes \eta, \tag{2.11}$$

where $X \in \Gamma(TM)$ and $\mathcal{L}_X g$ is the Lie-derivative of g along X given by:

$$(\mathcal{L}_X g)(Y,Z) = g(\nabla_Y X, Z) + g(\nabla_Z X, Y) \tag{2.12}$$

for all $Y, Z \in \Gamma(TM)$, and $a_1, a_2, \lambda \in R$. Equation (2.11), is a generalization of

(i) Killing's equation($a_1 = a_2 = \lambda = 0$),

(ii) Equation for homotheties ($a_1 = a_2 = 0$),

(iii) Ricci soliton ($a_1 = 0, a_2 = -1$)'

(iv) Case of Einstein-Weyl ($a_1 = 1, a_2 = \frac{-1}{n-2}$),

(v) Metric projective structures with skew-symmetric Ricci tensor in projective class ($a_1 = 1, a_2 = \frac{-1}{n-2}, \lambda = 0$),

(vi) Vacuum near-horizon geometry equation ($a_1 = 1, a_2 = \frac{1}{2}$), (*see* [5], [14], [15]).

Equation (2.11), is also a generalization of Einstein manifolds [9]. Note that, if $X = grad\psi$, where $\psi \in C^\infty(M)$, the generalized η-Ricci soliton equation is given by:

$$Hess\psi = -a_1 d\psi \odot d\psi + a_2 S + \lambda g + \mu\eta \otimes \eta. \tag{2.13}$$

3. Generalized η-Ricci solitons on $(M^{2n+1}, \phi, \xi, \eta, g)$

In a $(2n+1)$-dimensional trans Sasakian manifold M, we have the following relations:

$$R(X,Y)\xi = (\alpha^2 - \beta^2)[\eta(Y)X - \eta(X)Y] + 2\alpha\beta[\eta(Y)\phi X - \eta(X)\phi Y] \tag{3.1}$$
$$+[(Y\alpha)\phi X - (X\alpha)\phi Y + (Y\beta)\phi^2 X - (X\beta)\phi^2 Y]$$
$$S(X,\xi) = [(2n)(\alpha^2 - \beta^2) - (\xi\beta)]\eta(X) - ((\phi X)\alpha) - (2n-1)(X\beta), \tag{3.2}$$
$$Q\xi = ((2n)(\alpha^2 - \beta^2) - (\xi\beta))\xi + \phi(grad\alpha) - 2n(grad\beta), \tag{3.3}$$

where R is curvature tensor, while Q is the Ricci operator given by $S(X,Y) = g(QX,Y)$. Further in an trans Sasakian manifold , we have

$$\phi(grad\alpha) = 2n(grad\beta), \tag{3.4}$$

and

$$2\alpha\beta + (\xi\alpha) = 0. \tag{3.5}$$

Using (3.4) and (3.5), for constants α and β , we have

$$R(\xi,X)Y = (\alpha^2 - \beta^2)[g(X,Y)\xi - \eta(Y)X] \tag{3.6}$$
$$R(X,Y)\xi = (\alpha^2 - \beta^2)[\eta(Y)X - \eta(X)Y] \tag{3.7}$$
$$\eta(R(X,Y)Z) = (\alpha^2 - \beta^2)[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)] \tag{3.8}$$
$$S(X,\xi) = [2n(\alpha^2 - \beta^2)]\eta(X) \tag{3.9}$$
$$Q\xi = [2n(\alpha^2 - \beta^2)]\xi. \tag{3.10}$$

An important consequence of (2.4) is that ξ is a geodesic vector field, i.e., :

$$\nabla_\xi \xi = 0. \tag{3.11}$$

For arbitrary vector field X, we have that

$$d\eta(\xi, X) = 0. \tag{3.12}$$

The ξ-sectional curvature K_ξ of M is the sectional curvature of the plane spanned by ξ and a unit vector field X. From (3.7) we have

$$K_\xi = g(R(\xi,X),\xi,X) = (\alpha^2 - \beta^2). \tag{3.13}$$

It follows from (3.13) that ξ-sectional curvature does not depend on X.

Now, we are interested to prove the following theorem, which is the main result of this paper.

Theorem 3.1. *Let M be a trans-Sasakian manifold of dimension $(2n+1)$. If the generalized η-Ricci soliton satisfies (2.13) with the condition $a_1[\lambda + \mu + 2na_2(\alpha^2 - \beta^2)] \neq -(\alpha^2 - \beta^2)$, then ψ is a constant function. Furthermore, if $a_2 \neq 0$, then M is an Einstein manifold.*

From Theorem 3.1, we have the following remarks:

Remark 3.2. *If M is a trans-Sasakian manifold and the gradient Ricci soliton satisfies equation of $Hess\psi = -S + \lambda g$, then ψ is a constant function and M is an Einstein manifold.*

Remark 3.3. *In a trans-Sasakian manifold M, there is no non-constant smooth function ψ, such that $Hess\psi = \lambda g$, for some constant λ.*

For the proof of the Theorem 3.1, first we need to prove the following lemmas.

Lemma 3.4. *Let M be a trans-Sasakian manifold. Then we have*

$$(\mathcal{L}_\xi(\mathcal{L}_X g))(Y,\xi) = (\alpha^2 - \beta^2)g(X,Y) + g(\nabla_\xi\nabla_\xi X, Y) + Yg(\nabla_\xi X, \xi), \tag{3.14}$$

where $X, Y \in \Gamma(TM)$ and Y is orthogonal to ξ.

Proof. From the property of Lie-derivative we note that

$$(\mathcal{L}_\xi(\mathcal{L}_X g))(Y,\xi) = \xi((\mathcal{L}_X g)(Y,\xi)) - (\mathcal{L}_X g)(\mathcal{L}_\xi Y, \xi) - (\mathcal{L}_X g)(Y, \mathcal{L}_\xi \xi), \tag{3.15}$$

since $\mathcal{L}_\xi Y = [\xi, Y]$, $\mathcal{L}_\xi \xi = [\xi, \xi]$, by using (2.12) and (3.15), we have

$$\begin{aligned}(\mathcal{L}_\xi(\mathcal{L}_X g))(Y,\xi) &= \xi g(\nabla_Y X, \xi) + \xi g(\nabla_\xi X, Y) - g(\nabla_{[\xi,Y]} X, \xi) \\ &\quad - g(\nabla_\xi X, [\xi, Y]) \\ &= g(\nabla_\xi \nabla_Y X, \xi) + g(\nabla_Y X, \nabla_\xi \xi) + g(\nabla_\xi \nabla_\xi X, Y) \\ &\quad + g(\nabla_\xi X, \nabla_\xi Y) - g(\nabla_\xi X, \nabla_\xi Y) - g(\nabla_{[\xi,Y]} X, \xi) + g(\nabla_\xi X, \nabla_Y \xi),\end{aligned} \tag{3.16}$$

from (2.4), we get $\nabla_\xi \xi = -\alpha(\phi\xi) = 0$, so that we get

$$\begin{aligned}(\mathcal{L}_\xi(\mathcal{L}_X g))(Y,\xi) &= g(\nabla_\xi \nabla_Y X, \xi) + g(\nabla_\xi \nabla_\xi X, Y) - g(\nabla_{[\xi,Y]} X, \xi) \\ &\quad + Yg(\nabla_\xi X, \xi) - g(\nabla_Y \nabla_\xi X, \xi),\end{aligned} \tag{3.17}$$

using (2.6) and (3.17), we obtain

$$(\mathcal{L}_\xi(\mathcal{L}_X g))(Y,\xi) = g(R(\xi, Y)X, \xi) + g(\nabla_\xi \nabla_\xi X, Y) + Yg(\nabla_\xi X, \xi). \tag{3.18}$$

Now from (3.6), with $g(Y, \xi) = 0$, we get

$$g(R(\xi, Y)X, \xi) = g(R(X, \xi)\xi, Y) = (\alpha^2 - \beta^2)g(X, Y). \tag{3.19}$$

The Lemma follows from (3.18) and (3.19).

Now, we have another useful lemma

Lemma 3.5. *Let M be a Riemannian manifold, and let $\psi \in C^\infty(M)$. Then we have*

$$(\mathcal{L}_\xi(d\psi \odot d\psi))(Y, \xi) = Y(\xi(\psi))\xi(\psi) + Y(\psi)\xi(\xi(\psi)), \tag{3.20}$$

where $\xi, Y \in \Gamma(TM)$.

Proof. We calculate:

$$(\mathcal{L}_\xi(d\psi \odot d\psi))(Y,\xi) = \xi(Y(\psi)\xi(\psi) - [\xi, Y](\psi)\xi(\psi) - Y(\psi)[\xi,\xi](\psi)$$
$$= \xi(Y(\psi))\xi(\psi) + Y(\psi)\xi(\xi(\psi)) - [\xi, Y](\psi)\xi(\psi),$$

since $[\xi, Y](\psi) = \xi(Y(\psi)) - Y(\xi(\psi))$, we get

$$(\mathcal{L}_\xi(d\psi \odot d\psi))(Y,\xi) = [\xi, Y](\psi)\xi(\psi) + Y(\xi(\psi))\xi(\psi) + Y(\psi)\xi(\xi(\psi)) - [\xi, Y](\psi)\xi(\psi)$$
$$= Y(\xi(\psi))\xi(\psi) + Y(\psi)\xi(\xi(\psi)).$$

Lemma 3.6. *Let M be a trans-Sasakian manifold of dimension $(2n+1)$, and the generalized η-Ricci soliton satisfies equation (2.13). Then we have*

$$\nabla_\xi \; grad\psi = [\lambda + \mu + 2na_2(\alpha^2 - \beta^2)]\xi - a_1\xi(\psi)grad\psi. \tag{3.21}$$

Proof. Let $Y \in \Gamma(TM)$, from the definition of Ricci curvature S (2.7) , and the curvature condition (3.7), we have

$$\begin{aligned} S(\xi, Y) &= g(R(\xi, e_i)e_i, Y) \\ &= g(R(e_i, Y)\xi, e_i) \\ &= (\alpha^2 - \beta^2)[\eta(Y)g(e_i, e_i) - \eta(e_i)g(Y, e_i)] \\ &= (\alpha^2 - \beta^2)[(2n+1)\eta(Y) - \eta(Y)] \\ &= 2n(\alpha^2 - \beta^2)\eta(Y) \\ &= 2n(\alpha^2 - \beta^2)g(\xi, Y), \end{aligned}$$

where $\{e_1, e_2, ...e_i\}$, and $1 \le i \le 2n+1$ is an orthonormal frame on M, implies that:

$$\lambda g(\xi, Y) + \mu g(\xi, Y) + a_2S(\xi, Y) = \lambda g(\xi, Y) + \mu g(\xi, Y) + a_2 2n(\alpha^2 - \beta^2)g(\xi, Y) \tag{3.22}$$
$$= [\lambda + \mu + 2na_2(\alpha^2 - \beta^2)]g(\xi, Y).$$

From (2.13) and (3.22), we obtain

$$(Hess\psi)(\xi, Y) = -a_1\xi(\psi)(Y)(\psi) + [\lambda + \mu + 2na_2(\alpha^2 - \beta^2)]g(\xi, Y) \tag{3.23}$$
$$= -a_1\xi(\psi)g(grad\psi, Y) + [\lambda + \mu + 2na_2(\alpha^2 - \beta^2)]g(\xi, Y),$$

the Lemma follows from equation (3.23) and the definition of Hessian (2.9).

Now, with help of Lemma 3.4 , Lemma 3.5 and Lemma 3.6 we can prove the Theorem 3.1 .

Proof. Theorem 3.1. Let $Y \in \Gamma(TM)$, such that $g(\xi, Y) = 0$, from Lemma 3.4, with $X = grad\ \psi$, we have

$$2(\mathcal{L}_\xi(Hess\psi))(Y,\xi) = (\alpha^2 - \beta^2)Y(\psi) + g(\nabla_\xi\nabla_\xi \; grad\psi, Y) + Yg(\nabla_\xi \; grad\psi, \xi), \tag{3.24}$$

from Lemma 3.6, and equation (3.24), we get

$$2(\mathcal{L}_\xi(Hess\psi))(Y,\xi) = (\alpha^2 - \beta^2)Y(\psi) + [\lambda + \mu + 2na_2(\alpha^2 - \beta^2)]g(\nabla_\xi\xi, Y) - a_1g(\nabla_\xi(\xi(\psi)grad\ \psi), Y)$$

$$+[\lambda+\mu+2na_2(\alpha^2-\beta^2)]Yg(\xi,\xi)-a_1Y(\xi(\psi)^2), \tag{3.25}$$

since $\nabla_\xi\xi=0$ and $g(\xi,\xi)=1$, from equation (3.25), we obtain:

$$2(\mathcal{L}_\xi(Hess\psi))(Y,\xi)=(\alpha^2-\beta^2)Y(\psi)-a_1\xi(\xi(\psi))Y(\psi)-a_1\xi(\psi)g(\nabla_\xi\ grad\psi,Y) \tag{3.26}$$

$$-2a_1\xi(\psi)Y(\xi(\psi)).$$

From Lemma 3.6, equation (3.26), and since $g(\xi,Y)=0$, we have

$$2(\mathcal{L}_\xi(Hess\psi))(Y,\xi)=(\alpha^2-\beta^2)Y(\psi)-a_1\xi(\xi(\psi))Y(\psi)+a_1^2\xi(\psi)^2Y(\psi) \tag{3.27}$$

$$-2a_1\xi(\psi)Y(\xi(\psi)).$$

Note that, from (2.4) and (2.5), we have $\mathcal{L}_\xi g=0$, which is a Killing vector field, implies that $\mathcal{L}_\xi S=0$, taking the Lie derivative of the generalized η-Ricci soliton equation (2.13) yields

$$(\alpha^2-\beta^2)Y(\psi)-a_1\xi(\xi(\psi))Y(\psi)+a_1^2\xi(\psi)^2Y(\psi)-2a_1\xi(\psi)Y(\xi(\psi)) \tag{3.28}$$

$$=-2a_1Y(\xi(\psi))\xi(\psi)-2a_1Y(\psi)\xi(\xi(\psi)),$$

is equivalent to

$$Y(\psi)[(\alpha^2-\beta^2)+a_1\xi(\xi(\psi))+a_1^2\xi(\psi)^2]=0, \tag{3.29}$$

according to Lemma 3.6, we have

$$a_1\xi(\xi(\psi))=a_1\xi g(\xi,grad\ \psi) \tag{3.30}$$

$$=a_1g(\xi,\nabla_\xi\ grad\psi)$$
$$=a_1[\lambda+\mu+2na_2(\alpha^2-\beta^2)]-a_1^2\xi(\psi)^2,$$

by equations (3.29) and (3.30), we obtain

$$Y(\psi)[(\alpha^2-\beta^2)+a_1(\lambda+\mu+2na_2(\alpha^2-\beta^2))]=0, \tag{3.31}$$

since $a_1[\lambda+\mu+2na_2(\alpha^2-\beta^2)]\neq-(\alpha^2-\beta^2)$, we find that $Y(\psi)=0$, i.e., $grad\psi$ is parallel to ξ. Hence $grad\ \psi=0$ as $D=ker\eta$ is not integrable any where, which means ψ is a constant function.

Now, for particular values of α and β we have following cases:
For $\alpha=0$ or ($\beta=1$), we can state:

Corollary 3.7. *Let M be a β-Kenmotsu (or Kenmotsu) manifold of dimension n, and the generalized η-Ricci soliton satisfies (2.13) with condition $a_1[\lambda+\mu-2na_2\beta^2)]\neq\beta^2$ (or 1). Then ψ is a constant function. Furthermore, if $a_2\neq0$, then M is an Einstein manifold.*

For $\beta=0$, or ($\alpha=1$) we can state the following:

Corollary 3.8. *Let M be a α-Sasakian (or Sasakian) manifold of dimension n, and the generalized η-Ricci soliton satisfies (2.13) with condition $a_1[\lambda+\mu+2na_2\alpha^2)]\neq-\alpha^2$ (or -1). Then ψ is a constant function. Furthermore, if $a_2\neq0$, then M is an Einstein manifold.*

Acknowledgment. The author is thankful to the referee's for their valuable comments and suggestions towards the improvement of the paper.

References

[1] Blair, D. E. and Oubina, J. A. (1990). Conformal and related changes of metric on the product of two almost contact metric manifolds, Publ. Mat. 34: 199-207.

[2] Blaga, A. M. (2016). η-Ricci solitons on Lorentzian para-Sasakian manifolds, Filomat, 30(2): 489-496.

[3] Blaga, A. M. (2015). η-Ricci solitons on para-Kenmotsu manifolds, Balkan J. Geom. Appl. 20: 1-13.

[4] Cho, J. T. and Kimura, M. (2009). Ricci solitons and Real hypersurfaces in a complex space form, Tohoku math.J. 61: 205-212.

[5] Chrusciel, P. T., Reall, H. S. and Tod, P. (2006). On non-existence of static vacuum black holes with degenerate components of the event horizon, Classical Quantum Gravity, 23(2): 549-554.

[6] Calin, C. and Crasmareanu, M. (2012). η-Ricci solitons on Hopf Hypersurfaces in complex space forms, Rev. Roumaine Math. Pures Appl. 57(1): 55-63.

[7] Cao, H. D. (2009). Recent progress on Ricci solitons, Adv. Lect. Math. (ALM), 11: 1-38

[8] Case, J. S. Shu, Y., Wei, G. (2011). Rigidity of quasi Einstein metrics, Diff. Geo. Appl. 20: 93-100,

[9] Catino, G. (2012). Generalized quasi Einstein manifolds with harmonic Weyl tensor, Math. Z. 271: 751-756.

[10] Chinea, D., Gonzales, C. (1990). A classification of almost contact metric manifolds, Ann. Mat. Pura Appl. 156: 15-30.

[11] Friedan, D. (1985). Non linear models in $2 + \epsilon$ dimensions, Ann. Phys. 163: 318-419.

[12] Hamilton, R. S. (1988). The Ricci flow on surfaces, Mathematics and general relativity, (Santa Cruz. CA, 1986), Contemp. Math. 71, Amer. Math. Soc. 237-262.

[13] Gray, A. and Harvella, L. M. (1980). The sixteen classes of almost Hermitian manifolds and their linear invariants, Ann. Mat. Pura Appl., 123(4): 35-58.

[14] Jezierski, J. (2009). On the existence of Kundt's metrics and degenerate (or extremal) Killing horizons. Classical and Quantum Gravity, 26(3): 035011.

[15] Kunduri, H. K and Lucietti, J. (2013). Classification of near-horizon geometries of extremal black holes, Living Rev. Relativity, 16(8): 29-33.

[16] Kenmotsu, K. (1972). A class of almost contact Riemannian manifolds, Tohoku Math. J. 24(2): 93-103.

[17] Levy, H. (1925). Symmetric tensors of the second order whose covariant derivatives vanish, Ann. Math. 27(2): 91-98.

[18] Marrero, J. C. (1992). The local structure of Trans-Sasakian manifolds, Annali di Mat. Pura ed Appl. 162: 77-86.

[19] Mekki, El. A. M. and Cherif, A. M. (2017). Generalized Ricci Solitons on Sasakian anifolds, Kyungpook Math. J. 57: 677-682.

[20] Nurowski, P. and Randall, M. (2016). Generalized Ricci solitons, J. Geom. Anal. 26: 1280-1345.

[21] Nagaraja, H.G. and Premalatha, C. R. (2012). Ricci solitons in Kenmotsu manifolds, J. Math. Anal. 3(2): 18-24.

[22] Oubina, J. A. (1985). New classes of almost contact metric structures, Publ. Math. Debrecen 32: 187-193.

[23] Prakasha, D. G. and Hadimani, B. S. (2017). η-Ricci solitons on para-Sasakian manifolds. Journal of Geometry, 108(2), 383-392.

[24] Sharma, R. (2008). Certain results on K-contact and (k,μ)-contact manifolds, J. Geom. 89(1-2): 138-147.

[25] Siddiqi, M. D. (2018). Generalized Ricci Solitons on Trans-Sasakian Manifolds, Khayyam Journal of Mathematics, 4(2): 178-186.

[26] Siddiqi, M. D. (2018). Conformal η-Ricci solitons in δ-Lorentzian Trans-Sasakian manifolds, Int. Journal of Maps in Mathematics -IJMM., 1: 15-34.

[27] Turan, M., De, U. C. and Yildiz, A. (2012). Ricci solitons and gradient Ricci solitons on 3-dimensional trans-Sasakian manifolds, Filomat, 26(2): 363-370.

[28] Tripathi, M. M. (2008). Ricci solitons in contact metric manifolds. arXiv preprint arXiv:0801.4222.

[29] Tano, S. (1969). The automorphism groups of almost contact Riemannian manifolds,Tohoku Math.J. 21: 21-38.

[30] Vinu, K. and Nagaraja, H. G. (2017). η-Ricci solitons in trans-Sasakian manifolds, Commun. Fac. sci. Univ. Ank. Series A1, 66(2): 218-224.