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UU Group Rings

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Abstract. A ring is called *UU* if each its unit is a unipotent. We prove that the group ring $R[G]$ is a commutative UU ring if, and only if, R is a commutative UU ring and G is an abelian 2-group. This extends a result due to McGovern-Raja-Sharp (J. Algebra Appl., 2015) established for commutative nil-clean group rings. In some special cases we also discover when $R[G]$ is a non-commutative UU ring as our results are closely related to those obtained by Koşan-Wang-Zhou (J. Pure Appl. Algebra, 2016) and Sahinkaya-Tang-Zhou (J. Algebra Appl., 2017).
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1. Introduction and Fundamentals

Throughout the present note our notations are in agreement with [6], [7] and [9]. For instance, as usual, the symbol $R[G]$ stands for the group ring of an arbitrary group G over an arbitrary associative unital ring R, and $\omega(R[G])$ is its augmentation ideal. The used terminology is mainly standard as the new notions are stated explicitly below.

Imitating [3], we state the following:

Definition 1.1. A ring R is said to be *UU* if its unit group $U(R)$ satisfies the equality $U(R) = 1 + Nil(R)$, where $Nil(R)$ is the set of all nilpotent elements of R.

However, this definition is rather clumsy for applications and so the next necessary and sufficient condition from [3] will be useful in the sequel.

Proposition 1.2. *A ring R is UU if, and only if, $2 \in Nil(R)$ and $U(R)$ is a 2-group.*

On the other hand, a ring R is called *nil-clean* if $R = Nil(R) + Id(R)$, where $Id(R)$ is the set of all idempotents in R, that is, for every $r \in R$ there exist $q \in Nil(R)$ and $e \in Id(R)$ such that $r = q + e$. If, in addition, $qe = eq$ holds, the nil-clean ring is called *strongly nil-clean.*

The following criterion, which was used in the proof of [8, Theorem 2.12], is independently proved in [3] and [5], respectively.

Proposition 1.3. *A ring R is strongly nil-clean if, and only if, the Jacobson radical $J(R)$ is nil and $R/J(R)$ is boolean.*

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Even much more, in [3] was showed that *a ring is strongly nil-clean exactly when it is nil-clean UU* which amounts to *a ring is strongly nil-clean uniquely when it is nil-clean and its unit group is a 2-group*. In order to simplify the proof of Theorem 2.12 from [8], we shall use this key assertion in what follows (e.g., in Corollary 2.4) without any concrete referring.

A brief history of the best known principal achievements on group rings over such rings is like this: In [4] was found a complete description when the group ring $R[G]$ is nil-clean. This was further expanded in the non-commutative case in both [5] and [8] to the classes of strongly nil-clean and nil-clean rings, respectively.

The leitmotif of this short article is to generalize the aforementioned results to the large class of UU rings as well as to give a more elementary and direct proof of a theorem from [5] (see [8, Theorem 2.12], too). It is worthwhile noticing that some partial statements on commutative group rings of UU rings are given in [2, Section 5].

2. Main Results

We recall that a group is said to be *locally finite* if each its finite subset generates a finite subgroup, that is, each its finitely generated subgroup is finite. These groups are necessarily torsion. A type of such groups are the so-called *locally normal* groups, that are groups for which every finite subset can be embedded in a finite normal subgroup. For torsion abelian groups this property is always fulfilled, whereas in the non-abelian case the situation is more delicate being the classical Burnside's problem solved in the negative.

Before proceed by proving our major assertion, we need the next pivotal instrument from [3].

• Let I be a nil-ideal of a ring R. Then R is UU precisely when R/I is UU.

Our chief statement is the following one:

Theorem 2.1. *Let G be a group and R a ring.*

(i) If $R[G]$ is UU, then R is UU and G is a 2-group.

(ii) If G is locally finite, then $R[G]$ is UU if, and only if, R is UU and G is a 2-group.

(iii) If H is a normal subgroup of G such that H is locally normal and if $R[G]$ is UU, then $R[G/H]$ is UU.

Proof. (i) According to Proposition 1.2, we know that 2 is nilpotent in $R[G]$ and $U(R[G])$ is a 2-group. It now follows immediately that 2 is nilpotent in R and that $U(R) \leq U(R[G])$ and $G \leq U(R[G])$ are both 2-groups. Again Proposition 1.2 applies to get that R is UU, as wanted.

(ii) In view of point (i), we need to show only the "if" part. To that goal, since G is locally finite, choosing $x \in \omega(R[G])$, we deduce that $x \in \omega(R[H])$ for some finite subgroup H of G. But it is well known that the ideal $\omega(R[H])$ is nilpotent and thus nil (see, e.g., [1]). Hence the element x is nilpotent, so that the ideal $\omega(R[G])$ is nil. Taking into account that $R[G]/\omega(R[G]) \cong R$ and the truthfulness of the bullet above, we are now done.

(iii) First of all, we observe that the following isomorphism of group rings

$$R[G]/(\omega(R[H]) \cdot R[G]) \cong R[G/H]$$

is fulfilled. We claim that the relative augmentation ideal $\omega(R[H]) \cdot R[G]$ of $R[G]$ is nil. In fact, since H is locally normal, each element z of this ideal is contained in the ideal $\omega(R[F]) \cdot R[G]$, where F is a finite normal subgroup of H and so normal in G. As above, it follows from [1, Theorem 9] that $\omega(R[F])$ is nilpotent whence so is $\omega(R[F]) \cdot R[G]$, because for any natural i the formula $(\omega(R[F]) \cdot R[G])^i = (\omega(R[F]))^i \cdot R[G]$ holds, taking into account that F is a normal subgroup of

G. That is why, z is a nilpotent and so the claim sustained. Furthermore, the bullet alluded to above allows us to deduce that $R[G/H]$ is a UU ring, as pursued.

It is worth to noticing that the claim in point (i) that G is a 2-group contrasts the comments before Proposition 2.9 in [8].

The next affirmation is an immediate consequence of the preceding theorem.

Corollary 2.2. *Suppose R is a ring and G is an abelian group. Then $R[G]$ is a UU ring if, and only if, R is a UU ring and G is a 2-group.*

We now come to the promised above generalization of the basic result in [4].

Corollary 2.3. *A group ring $R[G]$ is a commutative UU ring if, and only if, R is a commutative UU ring and G is an abelian 2-group.*

So, we are ready to provide a more transparent proof of the following fact from [5] commented above:

Corollary 2.4. *Suppose that R is an arbitrary ring and G is a locally finite group. Then $R[G]$ is strongly nil-clean if, and only if, R is strongly nil-clean and G is a 2-group.*

Proof. The necessity is well-known and trivial. As for the sufficiency, it follows directly from Theorem 2.1 and [8, Theorem 2.3].

We close the work with the following question of interest.

Problem 2.5. Does it follow that Theorem 2.1 remains true without the assumption that G is locally finite?

This query definitely will hold in the affirmative, provided that the next implication is valid: If R is a ring having $\text{char}(R) = 2$ and G is a finitely generated 2-group, then the augmentation ideal $\omega(R[G])$ of the group ring $R[G]$ is nil.

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