Primary hyperideals of multiplicative hyperrings

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Abstract. In this article, we study prime, primary, C-hyperideals of multiplicative hyperrings in the sense of Rota [14]. Prime hyperideal avoidance lemma was proved in [4]. We mainly study union of primary hyperideals in hyperring. Among many results in this study we give the primary hyperideal avoidance lemma. Also we determine all prime, primary, C-ideals of quotient hyperring.

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1. Introduction

The study of hyperstructures dates back to [8]. In 1934, during the 8\textsuperscript{th} congress of Scandinavian Mathematicians, F. Marty introduced hypergroups as a generalization of groups. So far, it has been a developing area and many mathematicians has studied on hyperstructures. See [2,3,8,11]. One of the main reason for studying hyperstructures, it has a wide variety of applications to other areas such as geometry, lattices, automata, cryptography, coding theory, artificial intelligence and probabilities [3].

In [6], Krasner introduced the hyperrings in a study of approximations of complete valued fields. He defined \((R, +, \circ)\) as a hyperring with the following properties: (i) \((R, +)\) is a canonical hypergroup, (ii) \((R, \circ)\) is a semigroup and (iii) binary operation \(\circ\) is distributive over addition. The theory of hyperrings has been developed by many researchers. See [5-6,9,12-14]. There are various types of hyperrings and one of the important classes of hyperrings, called multiplicative hyperring, was introduced by Rota in [14]. Rota called \((R, +, \circ)\) a multiplicative hyperring if (i) \((R, +)\) is an abelian group, (ii) \((R, \circ)\) is a hypersemigroup, (iii) \(a \circ (b + c) \subseteq a \circ b + a \circ c\) and (iv) \(a \circ (-b) = -(a \circ b)\) for all \(a, b, c \in R\). Here, we mean a hypersemigroup by a nonempty set \(R\) with an associative hyperoperation \(\circ\), i.e,

\[
a \circ (b \circ c) = \bigcup_{t \in (b \circ c)} a \circ t = \bigcup_{s \in (a \circ b)} s \circ c = (a \circ b) \circ c
\]

for all \(a, b, c \in R\). Further, if \(R\) is a multiplicative hyperring with \(a \circ b = b \circ a\) for all \(a, b \in R\), then \(R\) is called a commutative multiplicative hyperring.

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Throughout, all hyperrings are assumed to be commutative hyperrings in the sense of Rota [14]. In particular, \( R \) always denotes such a hyperrings. For any two nonempty subsets \( K, L \) of \( R \), we define

\[
K \odot L = \bigcup_{k \in K,i \in L} k \odot i.
\]

Also by induction we can mention that product of finitely many nonempty subsets of \( R \). As well we denote the sum of the set \( a \odot b \) at \( k \) times by \( k(a \odot b) = \{x_1 + x_2 + \ldots + x_k : x_i \in a \odot b \text{ for } 1 \leq i \leq k\} \). Suppose that \( I \) is an additive subgroup of \( R \). If \( r \odot a \subseteq I \) for all \( r \in R \) and \( a \in I \), then \( I \) is called a hyperideal of \( R \). Let \( I \) be a hyperideal of \( R \). Then \( I \) is said to be a \( C \)-ideal if \( a_1 \odot a_2 \odot \ldots \odot a_n \cap I \neq \emptyset \), for \( n \in \mathbb{N}, a_i \in R, 1 \leq i \leq n \), imply that \( a_1 \odot a_2 \odot \ldots \odot a_n \subseteq I \) [4].

Prime ideal has a distinguished place in commutative algebra. It is useful tool for understanding ideal structure of rings. There are several studies on this issue. See [1,7]. Also, Primeness of hyperideals was first studied by Processi and Rota in [13]. Recall that a proper hyperideal \( P \) of \( R \) is said to be a prime hyperideal if the condition \( a \odot b \subseteq P \) implies either \( a \in P \) or \( b \in P \) [13]. After that, in [4] Dasgupta studied on prime and primary hyperideals in hyperrings. A proper hyperideal \( Q \) of \( R \) is called a primary hyperideal if \( a \odot b \subseteq Q \) then either \( a \in Q \) or \( b^n \subseteq Q \) for some \( n \in \mathbb{N} \) [4]. For any given hyperideal \( I \) of \( R \), \( \sqrt{I} \) denote the intersection of all prime hyperideals containing \( I \), if there is no such prime hyperideal of \( R \), we assume that \( \sqrt{I} = R \). Also the hyperideal \( \{r \in R : r^n \subseteq I \text{ for some } n \in \mathbb{N}\} \) will be designated by \( D(I) \) and note that the inclusion \( D(I) \subseteq \sqrt{I} \) always holds. In addition, if \( I \) is a \( C \)-ideal of \( R \), other inclusion holds by Proposition 3.2 of [4].

In this paper, our aim is to study union of primary hyperideals in hyperrings. Among many results in this paper we give primary hyperideal avoidance theorem with Theorem 2.6. Let \((R, +, \odot)\) and \((R', +, \odot)\) be two hyperrings and \( f : R \to R' \) a function such that \( f(a + b) = f(a) + f(b) \) and \( f(a \odot b) \subseteq f(a) \odot f(b) \) for all \( a, b \in R \). Then \( f \) is called a homomorphism. In particular \( f \) is called a good homomorphism in case \( f(a \odot b) = f(a) \odot f(b) \) [5]. Furthermore, the kernel of a homomorphism is defined as \( Ker(f) = f^{-1}(\{0\}) = \{r \in R : f(r) \in \{0\}\} \) and note that \( f(r) \) may not be a zero element. From Proposition 2.7 to Corollary 2.12 we investigate the prime, primary, \( C \)-hyperideals under good homomorphism and determine all prime, primary, \( C \)-hyperideal of any quotient hyperring. Finally we extend the primary hyperideal avoidance theorem to cosets of any quotient hyperring.

For more detail and terminology on hyperrings, the reader may consult [5].

2. Primary hyperideals in multiplicative hyperrings

Throughout, \( R \) denotes a commutative multiplicative hyperring.

**Definition 2.1.** Let \( Q \) be a proper hyperideal of \( R \). Then \( Q \) is called a primary hyperideal of \( R \) if for each \( a, b \in R \) and whenever \( a \odot b \subseteq Q \) then either \( a \in Q \) or \( b^n \subseteq Q \) for some \( n \in \mathbb{N} \) [4].

**Lemma 2.2.** Suppose that \( Q \) is a primary \( C \)-hyperideal of \( R \). Then \( \sqrt{Q} \) is a prime \( C \)-hyperideal of \( R \).

**Proof.** By [4, Proposition 3.6] we know that \( \sqrt{Q} \) is a prime hyperideal. Now we show that \( \sqrt{Q} \) is a \( C \)-ideal of \( R \). Let \( a_1 \odot a_2 \odot \ldots \odot a_n \cap \sqrt{Q} \neq \emptyset \) for some \( a_1, a_2, \ldots, a_n \in R \). Then we have \( x \in a_1 \odot a_2 \odot \ldots \odot a_n \) such that \( x \in \sqrt{Q} \). This implies that \( x^t \subseteq (a_1 \odot a_2 \odot \ldots \odot a_n)^t \) and \( x^t \subseteq Q \) for some \( t \in \mathbb{N} \). Since \( Q \) is a \( C \)-ideal and \( (a_1 \odot a_2 \odot \ldots \odot a_n)^t \cap Q \neq \emptyset \), we get \( (a_1 \odot a_2 \odot \ldots \odot a_n)^t \subseteq Q \) and so \( (a_1 \odot a_2 \odot \ldots \odot a_n) \subseteq \sqrt{Q} \). Hence \( \sqrt{Q} \) is a \( C \)-ideal of \( R \).

In [2], author proved prime hyperideal avoidance theorem for \( C \)-ideals: Let \( P_1, P_2, \ldots, P_n \) be prime \( C \)-hyperideals such that \( I \subseteq P_1 \cup P_2 \cup \ldots \cup P_n \) then \( I \subseteq P_i \) for some \( 1 \leq i \leq n \). Actually
the condition of C-ideal is not necessary. Further if at most two of \( P_i \)’s are not prime hyperideal, then the result is also valid. Now we prove the primary avoidance theorem but first we give some results we need.

**Lemma 2.3.** Let \( Q \) be a proper hyperideal of \( R \). Then \( Q \) is primary hyperideal if and only if \( J \circ K \subseteq Q \) implies either \( J \subseteq Q \) or \( K \subseteq D(Q) \), where \( J, K \) are hyperideals of \( R \).

**Proof.** Suppose that \( Q \) is primary hyperideal of \( R \) such that \( J \circ K \subseteq Q \) and \( J \nsubseteq Q \). Then there exists an element \( j \notin Q \) for some \( j \in J \). Take any \( k \in K \). Then we have \( j \circ k \subseteq J \circ K \subseteq Q \). Since \( Q \) is primary hyperideal, we have \( k^n \subseteq Q \) and so \( k \in D(Q) \). This implies that \( K \subseteq D(Q) \). Conversely, let \( a \circ b \subseteq Q \) for some \( a, b \in R \). Then we have \( \langle a \circ b \rangle \subseteq Q \). Take any \( t \in \langle a \rangle \) and \( s \in \langle b \rangle \). Then

\[
t = \sum_{i=1}^{n_r} x_i + n'_i a \quad \text{for some } n'_i \in \mathbb{Z} \quad \text{and} \quad x_i \in r_i \circ a \quad \text{and also } \quad s = \sum_{i=1}^{n_s} y_i + s'_i a \quad \text{for some } s'_i \in \mathbb{Z} \quad \text{and} \quad y_i \in r'_i \circ b.
\]

This implies that \( t \circ s = \left( \sum_{i=1}^{n_r} x_i + n'_i a \right) \circ \left( \sum_{i=1}^{n_s} y_i + s'_i a \right) \subseteq \sum_{i=1}^{n_r} \sum_{j=1}^{n_s} x_i \circ y_j + n'_i s'_i \sum_{i=1}^{n_r} a \circ y_i + s'_i \sum_{i=1}^{n_s} x_i \circ b + n'_i s'_i (a \circ b) \subseteq \langle a \circ b \rangle \subseteq Q \) and so \( \langle a \rangle \circ \langle b \rangle \subseteq Q \). Then we have either \( \langle a \rangle \subseteq Q \) or \( \langle b \rangle \subseteq D(Q) \). Thus we get \( a \in Q \) or \( b^n \subseteq Q \) for some \( n \in \mathbb{N} \).

By induction hypothesis one can easily obtain following result:

**Corollary 2.4.** If \( Q \) is a primary hyperideal of \( R \) such that \( J_1 \circ J_2 \circ \ldots \circ J_n \subseteq Q \) then either \( J_i \subseteq Q \) or \( J_i \subseteq D(Q) \) for some \( 2 \leq i \leq n \).

Let \( I \subseteq Q_1 \cup \ldots \cup Q_n \) be a covering of hyperideals of \( R \). Then this covering is called efficient if none of the \( Q_i \)’s are superfluous. Note that a covering by two hyperideals can not be efficient.

**Proposition 2.5.** Suppose that \( I \subseteq Q_1 \cup \ldots \cup Q_n \) is an efficient covering of hyperideals of \( R \), where \( I \) is a hyperideal of \( R \). If \( \sqrt{Q_i} \nsubseteq \sqrt{Q_j} \) for each \( i \neq j \), then any of \( Q_i \)’s are not primary hyperideal of \( R \).

**Proof.** First we show that \( \sqrt{I} = \sqrt{D(J)} \) for any hyperideal \( J \) of \( R \). The inclusion \( \sqrt{I} \subseteq \sqrt{D(J)} \) always holds. Take a prime hyperideal \( P \) containing \( J \). Then it is sufficient to show that \( P \) contains \( D(J) \). Let \( x \in D(J) \). Then we have \( x^n = x \circ x \circ \ldots \circ x \subseteq J \subseteq P \) for some \( n \in \mathbb{N} \). This yields \( x \in P \) and thus \( \sqrt{D(J)} \subseteq \sqrt{I} \). Since covering is efficient, we have \( n > 2 \). Assume that \( Q_1 \) is a primary hyperideal of \( R \). Again since the covering is efficient, we have \( I \cap Q_2 \cap Q_3 \cap \ldots \cap Q_n \subseteq I \cap Q_1 \subseteq Q_1 \) by [10]. As \( I \nsubseteq Q_1 \) and \( I \circ Q_2 \circ \ldots \circ Q_n \subseteq Q_1 \), by previous corollary, there exists \( 2 \leq j \leq n \) such that \( Q_j \subseteq D(Q_1) \) and so \( \sqrt{Q_j} \subseteq \sqrt{D(Q_1)} \). This is a contradiction.

By the following theorem we can obtain prime hyperideal avoidance theorem in [4, Proposition 2.19] without assumption C-ideal.

**Theorem 2.6.** Suppose that \( I \subseteq Q_1 \cup \ldots \cup Q_n \) is a covering and at most two of \( Q_i \)’s are not primary hyperideal of \( R \). If \( \sqrt{Q_i} \nsubseteq \sqrt{Q_j} \) for each \( i \neq j \), then \( I \nsubseteq Q_i \) for some \( 1 \leq i \leq n \).

**Proof.** If \( n = 2 \), then the result is valid. Also we may assume that the covering is efficient so \( n \neq 2 \). Suppose \( n > 2 \). But in this case, there exists a primary hyperideal \( Q_j \) of covering and this contradicts by Proposition 2.5. So we have \( n = 1 \) and this completes the proof.

**Proposition 2.7.** Let \( f : R \to S \) be a good homomorphism and \( I, J \) be hyperideals of \( R \) and \( S \), respectively. Then the followings are satisfied:
Let Proposition 2.8.

(ii) If $J$ is a prime hyperideal of $S$, then $f^{-1}(J)$ is a prime hyperideal of $R$.

Proof. (i) It is easy to see that $f(I)$ is a hyperideal of $S$ since $f$ is good epimorphism. Let $f(a) \circ f(b) \subseteq f(I)$ for $a, b \in R$. Since $f$ is homomorphism, $f(a \circ b) \subseteq f(a) \circ f(b) \subseteq f(I)$. Now take any $t \in a \circ b$. Then we have $f(t) \subseteq f(a \circ b) \subseteq f(I)$ and so $f(t) = f(x)$ for some $x \in I$. This implies that $f(t - x) = 0 \in (I)$, that is, $t - x \in Ker(f) \subseteq I$ and so $t \in I$. Thus we conclude that $a \circ b \subseteq I$. Since $I$ is prime hyperideal of $R$, we get $a \in I$ or $b \in I$ and so either $f(a) \in f(I)$ or $f(b) \in f(I)$. Consequently $f(I)$ is prime hyperideal of $S$.

(ii) First note that $f^{-1}(J)$ is a hyperideal of $R$. Let $a \circ b \subseteq f^{-1}(J)$ for $a, b \in R$. Then we have $f(a \circ b) = f(a) \circ f(b) \subseteq J$. Since $J$ is prime hyperideal of $S$, we have $f(a) \in J$ or $f(b) \in J$ and so either $a \in f^{-1}(J)$ or $b \in f^{-1}(J)$.

Proposition 2.8. Let $f : R \to S$ be a good homomorphism and $I, J$ be hyperideals of $R$ and $S$, respectively. Then the followings are satisfied:

(i) If $I$ is a prime hyperideal containing $Ker(f)$ and $f$ is a epimorphsim, then $f(I)$ is a prime hyperideal of $S$.

(ii) If $I$ is a $C$-hyperideal of $S$, then $f^{-1}(I)$ is a $C$-hyperideal of $R$.

Proof. (i) Let $s_1 \circ s_2 \circ \ldots \circ s_n \cap f(I) \neq \emptyset$ for some $s_1, s_2, \ldots, s_n \in S$. Since $f$ is epimorphsim, we have $f(a_i) = s_i$ for some $a_i \in R, 1 \leq i \leq n$. Then we have $(f(a_1) \circ f(a_2) \circ \ldots \circ f(a_n)) \cap f(I) = f(a_1 \circ a_2 \circ \ldots \circ a_n) \cap f(I) \neq \emptyset$ because $f$ is good homomorphism. So there exists $t \in a_1 \circ a_2 \circ \ldots \circ a_n$ such that $f(t) \in f(I)$. Since $Ker(f) \subseteq I$, we conclude that $t \in I$ and so $a_1 \circ a_2 \circ \ldots \circ a_n \cap I \neq \emptyset$. As $I$ is a $C$-ideal of $R$, $a_1 \circ a_2 \circ \ldots \circ a_n \subseteq I$ and so $f(a_1) \circ f(a_2) \circ \ldots \circ f(a_n) = f(a_1 \circ a_2 \circ \ldots \circ a_n) \subseteq f(I)$ that is $s_1 \circ s_2 \circ \ldots \circ s_n \subseteq f(I)$.

(ii) Let $a_1 \circ a_2 \circ \ldots \circ a_n \cap f^{-1}(J) \neq \emptyset$ for some $a_1, a_2, \ldots, a_n \in R$. This implies that $t \in f^{-1}(J)$ for some $t \in a_1 \circ a_2 \circ \ldots \circ a_n$. Let $n \in J \cap f(a_1 \circ a_2 \circ \ldots \circ a_n)$. Then we have $J \cap (f(a_1) \circ f(a_2) \circ \ldots \circ f(a_n)) \neq \emptyset$. Since $J$ is a $C$-ideal of $S$, we conclude that $f(a_1) \circ f(a_2) \circ \ldots \circ f(a_n) = f(a_1 \circ a_2 \circ \ldots \circ a_n) \subseteq J$ and so $a_1 \circ a_2 \circ \ldots \circ a_n \subseteq f^{-1}(J)$.

Suppose that $J$ is a hyperideal of $R$. Then quotient abelian group $R/I = \{a + I : a \in R\}$ becomes a hyperping with the multipication $(a + I) \circ (b + I) = (a + I) \circ (b + I)$ for $a, b \in R$. In this case $R/I$ is called quotient hyperping. One can easily prove that all hyperideal of $R/I$ is of the form $J/I$, where $J$ is a hyperideal of $R$ containing $I$, since the natural homomorphism $\pi : R \to R/I, \pi(r) = r + I$ is a good epimorphism.

Proposition 2.9. Let $I \subseteq P$ be hyperideals of $R$. Then the followings are satisfied.

(i) $P$ is prime hyperideal of $R$ iff $P/I$ is a prime hyperideal of $R/I$. In particular all prime hyperideal of $R/I$ is of the form $P/I$, where $P$ is a prime hyperideal of $R$ containing $I$.

(ii) $P$ is $C$-hyperideal of $R$ iff $P/I$ is a $C$-hyperideal of $R/I$. In particular all $C$-hyperideal of $R/I$ is of the form $P/I$, where $P$ is a $C$-hyperideal of $R$ containing $I$.

Proof. (i) Consider the natural homomorphism $\pi : R \to R/I$ defined by $\pi(r) = r + I$. Since $f$ is good epimorphism, the claim follows from Proposition 2.7.
(ii) It follows from Proposition 2.8.

**Corollary 2.10.** Suppose that $I \subseteq J$ are hyperideals of $R$. Then the followings are satisfied:

(i) $\sqrt{J/I} = \sqrt{J}/I$.

(ii) $D(J/I) = D(J)/I$.

(iii) If $J$ is a C-hyperideal of $R$. Then $D(J/I) = \sqrt{J/I}$.

**Proof.** (i) It follows from Proposition 2.9.

(ii) Let $a + I \in D(J/I)$ for some $a \in R$. Then we have $(a + I)^n \subseteq J/I$. Take any $t \in a^n$. Since $t + I \in (a + I)^n$, we have $t + I \in J/I$ and so $t \in J$. Thus we have $a^n \subseteq J$ and this implies that $a + I \in D(J)/I$. Conversely assume that $a + I \in D(J)/I$ for $a \in R$. Then we have $a \in D(J)$ and so $a^n \subseteq J$ for some $n \in \mathbb{N}$. Take any $t + I \in (a + I)^n$, then we have $t + I = s + I$ for some $s \in a^n$. This implies that $t = s \in I \subseteq J$ and so $t = (t - s) + s \in I + a^n \subseteq J$ and thus $t + I \in J/I$. This implies that $(a + I)^n \subseteq (J/I)$ that is $a + I \in D(J/I)$.

(iii) It follows from [4, Proposition 3.2] and Proposition 2.9.

**Proposition 2.11.** Let $f : R \rightarrow S$ be a good homomorphism and $I, J$ be hyperideals of $R$ and $S$, respectively. Then the followings are satisfied:

(i) If $I$ is a primary hyperideal containing $\text{Ker}(f)$ and $f$ is an epimorphism, then $f(I)$ is a primary hyperideal of $S$.

(ii) If $J$ is a primary hyperideal of $S$, then $f^{-1}(J)$ is a primary hyperideal of $R$.

**Proof.** (i),(ii): The proof is similar to Proposition 2.8.

**Corollary 2.12.** Let $I \subseteq Q$ be hyperideals of $R$. Then,

(i) $Q$ is primary hyperideal of $R$ iff $Q/I$ is a primary hyperideal of $R/I$. In particular all primary hyperideal of $R/I$ is of the form $Q/I$, where $Q$ is a primary hyperideal of $R$ containing $I$.

(ii) Suppose that $Q$ is a primary C-hyperideal of $R$ containing $I$. Then $\sqrt{Q/I}$ is a prime $C$-hyperideal of $R/I$.

**Proof.** (i): Follows from Proposition 2.11.

(ii): Follows from (i), Lemma 2.2, Proposition 2.9 and Corollary 2.10.

**Corollary 2.13.** Suppose that $I$ is a hyperideal of $R$. Then primary hyperideal avoidance theorem holds for quotient hyperring $R/I$.

**Proof.** It follows from Corollary 2.12 and Theorem 2.6.

Now we extend the primary hyperideal avoidance theorem to cosets of any quotient hyperring. Suppose that $J, J_1, J_2, \ldots, J_n$ are hyperideals of $R$ and $a_i \in R$ for each $i = 1, 2, \ldots, n$. Then we say a covering of cosets $J \subseteq (J_1 + a_1) \cup (J_2 + a_2) \cup \ldots \cup (J_n + a_n)$ is efficient if no cosets is unnecessary. Now we take such an efficient covering of cosets and assume that $a_k = a$ for all $k = 1, 2, \ldots, n$. Then we obtain an efficient covering $J - a \subseteq J_1 \cup J_2 \cup \ldots \cup J_n$. If $n = 1$, then we have $J - a \subseteq J_1$ and so $J \subseteq J_1 + a$. Since $0 \in J$, we get $a \in J_1$, that is, $J \subseteq J_1$. 
Lemma 2.14. Suppose that $J \subseteq (J_1 + a_1) \cup (J_2 + a_2) \cup \ldots \cup (J_n + a_n)$ is an efficient covering of cosets, where $J, J_1, J_2, \ldots, J_n$ are hyperideals of $R$ and $a_i \in R$ for each $i = 1, 2, \ldots, n$. Then $J \cap \bigcap_{i \neq k} J_i \subseteq J_k$ but $J \not\subseteq J_k$.

Proof. It is similar to proof of Lemma 4 in [7].

Proposition 2.15. Suppose that $Q_1, Q_2, \ldots, Q_n$ are hyperideals of $R$ and $J$ is a hyperideal such that $J + a \subseteq Q_1 \cup Q_2 \cup \ldots \cup Q_n$ is an efficient covering. If $\sqrt{Q_i} \not\subseteq \sqrt{Q_k}$ for each $i \neq k$, then no $Q_i$ is primary hyperideal of $R$.

Proof. Assume that $Q_1$ is primary hyperideal of $R$. Then by previous lemma, we get $J \cap Q_2 \cap Q_3 \cap \ldots \cap Q_n \subseteq Q_1$ and also $J \not\subseteq Q_1$. Note that $J \circ Q_2 \circ \ldots \circ Q_n \subseteq J \cap Q_2 \cap Q_3 \cap \ldots \cap Q_n \subseteq Q_1$. Then by Corollary 2.4 we get $Q_i \subseteq D(Q_1)$ and so $\sqrt{Q_i} \subseteq \sqrt{Q_1}$ which is a contradiction.

Theorem 2.16. Suppose that $Q_1, Q_2, \ldots, Q_n$ are hyperideals and at most $n - 1$ of them are not primary hyperideal of $R$. If $\sqrt{Q_i} \not\subseteq \sqrt{Q_k}$ for $i \neq k$ and $J + a \subseteq Q_1 \cup Q_2 \cup \ldots \cup Q_n$, then $J + a \subseteq Q_i$. Further, $J + \langle a \rangle \subseteq Q_i$ for some $i \in \{1, 2, \ldots, n\}$.

Proof. We may assume that covering is efficient. If $n > 1$, there exists a primary C-hyperideal $Q_i$ of $R$ which contradicts by previous proposition. So we have $n = 1$, $J + a \subseteq Q_i$ for some $i \in \{1, 2, \ldots, n\}$. Since $0 \in J$, we have $a \in Q_i$ and so $\langle a \rangle \subseteq Q_i$. As $a \in Q_i$ and $J + a \subseteq Q_i$, we get $J \subseteq Q_i$ and hence $J + \langle a \rangle \subseteq Q_i$.

References


