Rings Whose Elements Satisfy Quadratic Equations

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1. Introduction and Fundamentals

Everywhere in the text of this paper, all our rings $R$ are assumed to be associative, containing the identity element 1, which differs from the zero element 0. Our terminology and notations are mainly in agreement with [8]. For instance, $J(R)$ denotes the Jacobson radical of $R$, and $Id(R)$ denotes the set all idempotents in $R$. We also recall that a ring $R$ is said to be Boolean if each its element is a solution of the equation $x^2 = x$.

A brief history of the investigated theme on certain quadratic equations in arbitrary rings and extensions of Boolean rings is as follows: Rings whose elements satisfy the equations $x^2 = x$ or $x^2 = -x$ have been characterized in [5]. It has been shown that such a ring is isomorphic to either a Boolean ring $B$, to $Z_3$, or to $B \times Z_3$. It is direct to see that $x^2 = x$ or $x^2 = -x$ imply $x^3 = x$. In this way, rings satisfying the equation $x^3 = x$ were examined in [6]; it was proved there that they are commutative being a subdirect product of copies of the fields $Z_2$ and $Z_3$, and also that their elements are sums of two (commuting) idempotents. In this direction, in [1] it was established that $x^3 = x$ is tantamount to $x^2 = xv$, where $v = x^2 + x - 1$ is an involution, i.e., $v^2 = 1$. And finally, in [3] were completely classified those rings whose elements are solutions of the equations $x^3 = x$ or $x^3 = -x$.

The leitmotif of the current article is to study a class of rings whose elements satisfy combinations of the equations $x^2 = x$, $x^2 = -x$, $x^2 = 2 + x$ and $x^2 = 2 - x$. This is a quite perspective extension of the classical Boolean rings and some interesting types of rings will occur here. All characterizations will be given up to an isomorphism.

Besides, we shall demonstrate an easier proof of well-known results related to this topic. Some of the results could be found in [2] as well, but however we shall state them here (with proofs) for the readers’ convenience and completeness of the exposition.

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2. Main Result and Questions

We first begin with a refinement of results from [6] and [7] providing more transparent proofs. Specifically, the following holds (see also [2]).

**Proposition 2.1.** For a ring $R$ the next four items are equivalent:

(i) All elements of $R$ satisfy the equation $x^3 = x$.

(ii) $R$ is a subdirect product of copies of the fields $\mathbb{Z}_2$ and $\mathbb{Z}_3$.

(iii) $R$ is commutative and each element of $R$ is a sum of two (commuting) idempotents.

(iv) $R$ is commutative and each element of $R$ is a difference of two (commuting) idempotents.

**Proof.** (i) $\Rightarrow$ (ii) and (i) $\Rightarrow$ (iii). Since $2^3 = 2$, we have that 6 = 0 and so the Chinese Remainder Theorem allows us to get the decomposition $R \cong R_1 \times R_2$, where $R_1$ is either zero or a ring of characteristic 2 and $R_2$ is either zero or a ring of characteristic 3. We claim that $R_1$ is Boolean. In fact, the equation $x^3 = x$ continue to hold in $R_1$ for any its element, so that $(x + 1)^3 = x + 1$. Consequently, $3x^2 + 3x = 0$ ensuring that $x^2 = x$, as needed. That is why, as it is well-known, $R_1$ is a subdirect product of copies of the field $\mathbb{Z}_2$. As for the ring $R_2$, the equation $x^3 = x$ again holds. So, for any its element $x$, one observes that $-x = (-x - x^2) + x^2$, where both $-x - x^2$ and $x^2$ are idempotents. But then, the element $-(x - 2)$ is also a sum of two idempotents, say $e_1$ and $e_2$, respectively. Finally, one sees that $x = (1 - e_1) + (1 - e_2)$, as needed. Moreover, $R_2$ is known to be commutative by the Jacobson Theorem for commutativity, and also being regular, it is a subdirect product of fields in which $x^3 = x$ remains true. Thus $x^3 = x$ reduces to $x^2 = 1$, i.e., to $(x - 1)(x + 1) = 0$ whence to either $x = -1$ or $x = 1$. Since $3 = 0$, a routine check shows that these fields are isomorphic to $\mathbb{Z}_3$, as needed. So, $R$ is commutative and each its element is a sum of two idempotents, because these two properties are valid simultaneously in both $R_1$ and $R_2$.

(ii) $\Rightarrow$ (i). Since both $\mathbb{Z}_2$ and $\mathbb{Z}_3$ are even perfect fields, that is, $\mathbb{Z}_2^2 = \mathbb{Z}_2$ and $\mathbb{Z}_3^3 = \mathbb{Z}_3$, item (i) is now obvious.

(iii) $\iff$ (iv). If an arbitrary $r \in R$ is presentable as $r \in Id(R) + Id(R)$, then again $r + 1 \in Id(R) + Id(R)$ and hence $r \in Id(R) - (1 - Id(R))$, as required. Conversely, if an arbitrary element $r \in R$ is of the form $r \in Id(R) - Id(R)$, then again $r - 1 \in Id(R) - Id(R)$ and so $r \in Id(R) + (1 - Id(R))$, as expected.

(iv) $\Rightarrow$ (i). For any $r \in R$ writing $r = e - f$ for some $e, f \in Id(R)$, it is plainly verified that $r^3 = (e - f)^3 = e^3 - 3e^2f + 3ef^2 - f^3 = e - f = r$, as asserted. This completes the proof.

**Remark 2.2.** According to the current proof, and especially to the obvious equivalence of points (iii) and (iv), Proposition 2.1 from [7] is superfluous.

The next technicality is useful (cf. [2] as well).

**Lemma 2.3.** If $R = R_1 \times R_2$ is a ring whose elements are (either) a sum or a difference of two commuting idempotents, then $R_1$ or $R_2$ is a ring whose elements are sums of two commuting idempotents.

**Proof.** As observed above in the previous proposition, each element is a sum of two commuting idempotents precisely when each element is a difference of two commuting idempotents. With
Proposition 2.4. ([2]) Let $R$ be a ring. Then the next three items are equivalent:

(a) For all $x \in R$: $x^2 = x$ or $x^2 = 2 + x$ with $4 = 0$.

(b) Every element of $R$ is (either) idempotent or minus idempotent or a sum of two commuting idempotents and $4 = 0$.

(c) $R/J(R)$ is a Boolean ring and either $J(R) = \{0\}$ or $J(R) = \{0, 2\}$.

Proof. (a) $\Rightarrow$ (b). Letting $y \in R$ be an element which is not an idempotent, i.e., $y^2 \neq y$, we hence have that $y^2 = 2 + y$. Consider the element $y + 1$. If $(y + 1)^2 = y + 1$, then $y^2 = -y$ and thus $y = -y^2$ is the idempotent $y^2$. But if $(y + 1)^2 = 2 + (y + 1)$, we then have that $y^2 + y - 2 = 0$. However, combining this with $y^2 - y - 2 = 0$, one infers that $2y = 0$. Multiplying $y^2 - y - 2 = 0$ by $y$, we get that $y^3 = y^2$ whence $y^2 = y^2$ showing that $y^2$ is an idempotent. We finally write $y = 2 - y^2 = 1 + (1 - y^2)$, that is a sum of two idempotents, as required.

It is worthwhile noticing that we have not used here the limitation $4 = 0$.

(b) $\Rightarrow$ (c). Since each element is obviously a sum or a difference of two idempotents, we then employ [7] to get the claim.

(c) $\Rightarrow$ (a). Clearly, $2 \in J(R)$ and $2^2 = 4 \in J(R)$. Hence $4 = 0$ or $4 = 2$, i.e., $4 = 0$ or $2 = 0$ holds. Moreover, for an arbitrary $r \in R$, we have $r^2 - r \in J(R)$ and so $r^2 = r$ or $r^2 - r = 2$, that is, $r^2 = 2 + r$, as stated. This completes the proof.

Remark 2.5. We shall also add here an alternative direct proof of the implication (b) $\Rightarrow$ (a) which raises some new approaches than the given ones so far: We first claim that $R$ is commutative. Indeed, $x^2 = \pm x$ yields $x^3 = x$, while $x = e + f$ for some commuting $e, f \in Id(R)$ implies $x - 1 = e = (1 - f)$ and thus $(x - 1)^2 = x - 1$. A variation of the significant Jacobson Theorem for commutativity now ensures that $R$ has to be commutative, as claimed.

We now assert that $J(R) = \{0, 2\}$ and, in particular, that for any $h \in Id(R)$: $2h = 0$ or $2h = 2$. To that aim, given an arbitrary $z \in J(R)$ we have $z^2 = z, z^2 = -z$ or $z^2 = e + f$ for $e, f$ as above. In the first two cases, $z(1 + z) = 0$ assures $z = 0$ because $1 + z$ is always invertible. If now $z = e + f$, then $z - 1 = e = (1 - f)$ whence it is easily checked that $(z - 1)^2 = z - 1$. But $z - 1$ is a unit, so that $(z - 1)^2 = 1$ forcing that $z^2 = 2z$ and that $2z^2 = 0$. On the other hand, $z^2 = (e + f)^2 = z + 2ef$ and thus $2z^2 = 2z = z^2 = 0$. Consequently, $z = 2ef = e + f$. Multiplying by $1 - f$ on the right, we deduce $e(1 - f) = 0$, i.e., $e = ef$. Similarly, multiplying by $1 - e$ again on the right, we derive $f(1 - e) = 0$, i.e., $f = fe$. Finally, $e = f$ and $z = 2e$. If $e \neq 0$ and $2e \neq 2, whence 2(1 - e) \neq 0$, we see that $J(eR) \neq 0$ and $J((1 - e)R) \neq 0$. Therefore, consulting with Proposition 2.1, we conclude that $eR$ and $(1 - e)R$ are rings whose elements are not the sum of two idempotents. But then $R \cong eR \times (1 - e)R$ does not have elements of the given kind, thus contradicting Lemma 2.3.

Hereafter, if $2 = 0$, we are finished. So, assume that $2 \neq 0$. Suppose also that $y \in R$ with $y^2 \neq y$, whence $y^2 = -y$ or $y = e + f$. Considering the element $y + 2$, we have three possible cases.
Case 1: \((y + 2)^2 = y + 2\). Thus \(y^2 = -3y - 2\) amounting to \(y^2 = y + 2\), so we are set.

Case 2: \((y + 2)^2 = -(y + 2) = -y - 2\). Thus \(y^2 = -y - 2\). If we combine this with \(y^2 = -y\), we get \(2 = 0\) contrary to our assumption. We, therefore, combine it with \(y = e + f\). Squaring this last relation leads to \(y^2 = y + 2ef\). Since \(ef \in \text{Id}(R)\), by what we have just shown above, either \(2ef = 0\) or \(2ef = 2\). In the first case, we get the contradictive \(y^2 = y\), so that what remains is \(y^2 = y + 2\), as needed. Notice that only the equality \(y = e + f\) is absolutely enough to get the desired equation.

Case 3: \(y + 2\) is a sum of two commuting idempotents. As in Case 2, one has that either \((y + 2)^2 = y + 2\) or that \((y + 2)^2 = (y + 2) + 2 = y\). Hence \(y^2 = y + 2\), as needed, or \(y^2 = y\) which is impossible.

Combining the previous proposition together with the corresponding results from [7], one obtains the following fact:

**Remark 2.6.** For a ring \(R\) in which \(4 = 0\) (or even for which \(2 \in J(R)\)), every element is a sum or a difference of two commuting idempotents exactly when every element is (either) idempotent or minus idempotent or the sum of two commuting idempotents. As already indicated above, this somewhat refines Theorem 4.3 in [7].

We also can recall the following:

**Theorem 2.7.** ([2]) Each element of a ring \(R\) satisfies (one of) the equations \(x^2 = x\) or \(x^2 = 2 + x\) if, and only if, exactly one of the next two items is valid:

1. \(R \cong B\) or \(R \cong \mathbb{Z}_3\) or \(R \cong B \times \mathbb{Z}_3\), where \(B\) is a Boolean ring.
2. \(R/J(R) \cong B\) and either \(J(R) = \{0\}\) or \(J(R) = \{0, 2\}\), where \(B\) is a Boolean ring.

**Proof.** "\(\Rightarrow\)." Since these equations are trivially satisfied for the elements 0, 1 and 2, one writes that \(3^2 = 3\) or \(3^2 = 2 + 3\). So, \(6 = 0\) or \(4 = 0\). We thus come to two different basic cases:

Case: \(6 = 0\). Utilizing the Chinese Reminder Theorem, we write that \(R \cong R_1 \times R_2\), where \(R_1, R_2\) are rings of characteristic 2 and 3, respectively, whenever they are non-zero. Since \(2 = 0\) in \(R_1\), the equations \(x^2 = x\) and \(x^2 = 2 + x\) are equivalent. Thus \(R_1\) is Boolean.

As for the other direct factor \(R_2\), we consider the element \(x + 2\). If \((x + 2)^2 = x + 2\), then \(x^2 = 1\). Combined with \(x^2 = x\) this implies that \(x = 1\), whereas combined with \(x^2 = x + 2\) it implies that \(x = 2\). If now \((x + 2)^2 = 2 + (x + 2)\), then \(x^2 = 0\). Thus either \(x = 0\) or \(x + 2 = 0\) which yields that \(x = 1\), and it follows finally that either \(R_2 = \{0\}\) or \(R_2 = \{0, 1, 2\} \cong \mathbb{Z}_3\), as required.

Case: \(4 = 0\). It follows directly from Proposition 2.4.

"\(\Leftarrow\)." (1) A direct check shows that all of the elements in \(B, \mathbb{Z}_3\) and \(B \times \mathbb{Z}_3\) satisfy the desired two equations, so we omit the details.

(2) This condition follows directly by Proposition 2.4. This completes the proof.

The rings \(\mathbb{Z}_2, \mathbb{Z}_4, \mathbb{Z}_2 \times \mathbb{Z}_4\) and \(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4\) are concrete examples of rings for which condition (2) holds. However, \(\mathbb{Z}_4 \times \mathbb{Z}_4\) need not be of this type as simple calculations show.

**Remark 2.8.** Certainly, the theorem can also be written as follows: All elements of a ring \(R\) satisfy (one of) the equations \(x^2 = x\) or \(x^2 = 2 + x\) if, and only if, \(R \cong B\) or \(R \cong \mathbb{Z}_3\) or \(R \cong B \times \mathbb{Z}_3\) or \(R/J(R) \cong B\) with \(J(R) = \{0, 2\}\), where \(B\) is a Boolean ring.
Corollary 2.9. Each element of a ring $R$ satisfies the equations $x^2 = x$ or $x^2 = -x$ if, and only if, each element of $R$ satisfies the equations $x^2 = x$ or $x^2 = 2 + x$ and $6 = 0$.

Proof. If $2 = 0$, the claim follows, so we will assume that $2 \neq 0$.

$\Rightarrow$. Since $2^2 = -2$ (the other possibility $2^2 = 2$ is now unsuitable), we have $6 = 0$. Let $y \in R$ such that $y^2 \neq y$. Hence $y^2 = -y$. Consider the element $y + 2$. If $(y + 2)^2 = -y - 2$, then it follows that $y^2 + 5y + 6 = 0$, i.e., $y^2 = -5y = y$ which contradicts our assumption. We therefore may write $(y + 2)^2 = y + 2$ implying that $y^2 + 3y + 2 = 0$. Comparing this with $y^2 = -y$, one deduces that $-y = y + 2$ whence $y^2 = y + 2$, as desired.

$\Leftarrow$. Let $y \in R$ with $y^2 \neq y$. Thus $y^2 = 2 + y$. Consider the element $y + 4$. If $(y + 4)^2 = y + 4$, we obtain that $y^2 + 7y + 12 = 0$, that is, $y^2 = -7y = -y$ and we are set. Otherwise, if $(y + 4)^2 = 2 + (y + 4)$, we derive that $y^2 + 7y + 16 = 0$ amounting to $y^2 + y - 2 = 0$. Comparing that with $y^2 = 2 + y$, one concludes that $2y = 0$. Furthermore, multiplying both sides of $y^2 = y + 2$ by 2, it follows that $0 = 4$ which along with $6 = 0$ ensures that $2 = 0$, again contradicting to our assumption. This completes the proof.

Remark 2.10. The considered above equations $x^2 = x$ and $x^2 = 2 + x$ are equivalent to the equations $x^2 = -x$ and $x^2 = 2 - x$ via the substitution $x \rightarrow -x$. Moreover, it is reasonably adequate to consider the other combination $x^2 = x$ or $x^2 = 2 - x$ (being equivalent to the combination $x^2 = -x$ or $x^2 = 2 + x$ under the same substituting). However, interestingly, this does not give anything new, that is, such a ring is isomorphic to a Boolean ring, by the following arguments: we have $(-1)^2 = -1$ or $(-1)^2 = 2 - (-1)$. These yield that $2 = 0$, whence in both possibilities it must be that $x^2 = x$, as required.

With this critical observation in hand, we now arrive at our central result which unambiguously illustrates the extension of Boolean rings via quadratic equations as our accent here is to show how all the elements of some rings can be expressed in terms of quadratic equations.

Theorem 2.11. (1) All of the elements of a ring $R$ satisfy the equations $x^2 = x$ or $x^2 = -x$ or $x^2 = 2 + x$ if, and only if, these elements satisfy the equations $x^2 = x$ or $x^2 = 2 + x$.

(2) All of the elements of a ring $R$ satisfy the equations $x^2 = x$ or $x^2 = 2 - x$ or $x^2 = 2 + x$ if, and only if, $R \cong R_1 \times R_2 \times \mathbb{Z}_2$, and the elements in the rings $R_1, R_2$ satisfy the equations $x^2 = x$ or $x^2 = 2 + x$ with $4 = 0$ in $R_1$ and $3 = 0$ in $R_2$.

(3) All of the elements of a ring $R$ satisfy the equations $x^2 = x$ or $x^2 = 2 - x$ if, and only if, these elements satisfy the equations $x^2 = x$ or $x^2 = 2 + x$ or $x^2 = 2 - x$.

Proof. (1) Let we have the system of combinations $x^2 = x$ or $x^2 = -x$ or $x^2 = 2 + x$ which is equivalent to the system of combinations $x^2 = -x$ or $x^2 = x$ or $x^2 = 2 - x$ via the substitution $x \rightarrow -x$. We claim that the equality $x^2 = -x$ is redundant and so it can be omitted. In fact, the elements $0, 1, 2$ always satisfy these equations. However, one has that $3^2 = 3$ or $3^2 = -3$ or $3^2 = 2 + 3$. Thus $6 = 0$ or $12 = 0$ or $4 = 0$. In any case $12 = 0$. Therefore, by the Chinese Reminder Theorem, one may decompose $R \cong R_1 \times R_2$ for some rings $R_1, R_2$, where $2^2 = 0$ in $R_1$ and $3 = 0$ in $R_2$. Clearly, for all of the elements in both $R_1$ and $R_2$ these three equations remain valid. Firstly, we consider $R_1$ in which $4 = 0$: We assert that $\forall x \in R_1$ it must be that $x^2 = x$ or $x^2 = 2 + x$. Certainly, one assumes that $2 \neq 0$ as for otherwise the assertion sustained immediately. Suppose there exists $y \in R_1$ with $y^2 \neq y$ and $y^2 \neq 2 + y$, that is, $y^2 = -y$. We have three possibilities for the element $1 - y \in R_1$. If $(1 - y)^2 = 1 - y$, we obtain $y^2 = y$ which is impossible. If now
(1\!-\!y)^2 = (1\!-\!y)\), we derive that \(y^2 = 2 - y\) as \(4 = 0\), which combined with the remaining equality \(y^2 = -y\) assures that \(2 = 0\), which is against our assumption. If finally \((1\!-\!y)^2 = 2 + (1\!-\!y)\), we deduce then that \(y^2 = y + 2\) because \(4 = 0\), which is impossible too. Secondly, we consider \(R_2\) in which \(3 = 0\): As in the previous considerations, we just need to consider the element \(z \in R_2\) with \((1\!-\!z)^2 = -(1\!-\!z)\) which gives that \(z^2 = 1\) as \(3 = 0\). This, in combination with \(z^2 = -z\), ensures that \(z = -1\). So, \(z^2 = 2 + z\) and we are done.

(2) Let us have the system of combinations \(x^2 = x\) or \(x^2 = 2 - x\) or \(x^2 = 2 + x\) which is tantamount to the system of combinations \(x^2 = -x\) or \(x^2 = 2 + x\) or \(x^2 = 2 - x\) by the substitution \(x \rightarrow -x\). In fact, the elements 0, 1, 2 always satisfy these equations. However, one has that \(3^2 = 3\) or \(3^2 = 2 - 3\) or \(3^2 = 2 + 3\). Thus \(6 = 0\) or \(10 = 0\) or \(4 = 0\). In any case \(60 = 0\). Consequently, with the Chinese Reminder Theorem at hand, one may decompose \(R \cong R_1 \times R_2 \times R_3\) for some rings \(R_1, R_2, R_3\), where \(2^2 = 0\) in \(R_1\), \(3 = 0\) in \(R_2\) and \(5 = 0\) in \(R_3\). Evidently, in these three rings, which we shall now consider separately, these three equalities continue to hold. We claim that the equality \(x^2 = 2 - x\) is somewhat superfluous in both \(R_1\) and \(R_2\), and so it can be removed. In fact, we firstly consider \(R_1\): If for every \(x \in R_1\) we take the element \(x + 2\), we will get that \((x + 2)^2 = x + 2\) implies \(x^2 = x + 2\), \((x + 2)^2 = 2 + (x + 2)\) implies \(x^2 = x\) and \((x + 2)^2 = 2 - (x + 2)\) implies \(x^2 = -x\). Secondly, we consider \(R_2\): If for each \(y \in R_2\) we again consider the element \(y + 2\), we will get that \((y + 2)^2 = y + 2\) implies \(y^2 = 1\). This along with \(y^2 = y\) yields that \(y = 1\) and so \(y^2 = y\). With \(y^2 = y + 2\) it yields that \(y = -1\) or 2, and with \(y^2 = 2 - y\) it forces that \(y = 1\) whence again \(y^2 = y\). Next, \((y + 2)^2 = 2 - (y + 2)\) amounts to \(y^2 = y + 2\), as expected. Finally, \((y + 2)^2 = 2 + (y + 2)\) enables us that \(y^2 = 0\) which in combination with the existing three equalities \(y^2 = y\), \(y^2 = 2 - y\) and \(y^2 = 2 + y\) allows us to infer that \(y^2 = y\) or that either \(y = -1\) or \(y = 1\) implying that \(1 = 0\) which is wrong. We thus apply (1) to conclude our claim. Thirdly, we consider \(R_3\): If for any \(z \in R_3\) we also consider the element \(z + 2\), we will get that \((z + 2)^2 = z + 2\) equals to \(z^2 = 2z - 2\) which combined with the existing equations \(z^2 = z\), \(z^2 = 2 - z\) and \(z^2 = 2 + z\) gives that either \(1 = 0\), that is false, or that \(z = 3\) and so \(z^3 = -z\), or that \(z = -1\) and so \(z^3 = z\). If now \((z + 2)^2 = 2 + (2 + z)\), we obtain that \(z^2 = 2z\) and thus \(z^3 = 2z^2 = 4z = -z\) since \(5 = 0\), and finally \((z + 2)^2 = 2 - (z + 2)\) insures that \(z^4 = 1\) and hence \(z^3 = z\), as required. This substantiates our claim. We, therefore, can conclude according to [3] that \(R_3 \cong \mathbb{Z}_5\).

(3) Suppose now that we have the system of combinations \(x^2 = x\) or \(x^2 = -x\) or \(x^2 = 2 + x\) or \(x^2 = 2 - x\). We claim that the equation \(x^2 = -x\) can be dropped. In fact, as above, the elements 0, 1, 2 satisfy these equalities. As for the element 3, we must have that \(3^2 = 3\), or \(3^2 = -3\), or \(3^2 = 2 + 3\), or \(3^2 = 2 - 3\). Thus \(6 = 0\) or \(12 = 0\) or \(4 = 0\) or \(10 = 0\) and, resuming, one has that \(60 = 2^2 \cdot 3 \cdot 5\). Then the Chinese Reminder Theorem applies to write that \(R \cong R_1 \times R_2 \times R_3\) for some rings \(R_1, R_2, R_3\), where same as above \(2^2 = 0\) in \(R_1\), \(3 = 0\) in \(R_2\) and \(5 = 0\) in \(R_3\). Apparently, in these three rings, which we shall now consider separately, the four equalities remain fulfilled. Firstly, we consider \(R_1\): In contrast to above, we shall here consider the element \(x^2 = -x\). If \((x^2 - x)^2 = x^2 - x\), then \(x^4 - 2x^3 + x = 0\). Combining this with \(x^2 = -x\), we routinely obtain that \(x^2 = x\), because \(x = x^4\) and \(x^2 = x^4\). If next \((x^2 - x)^2 = x - x^2\), then \(x^4 - 2x^3 + 2x^2 - x = 0\) and the combination with \(x^2 = -x\) again gives that \(x^2 = x\). Furthermore, if \((x^2 - x)^2 = 2 + (x^2 - x)\), then \(x^4 - 2x^3 + x + z = 2 = 0\) and along with \(x^2 = -x\) we detect that \(x^2 = 2 + x\). Finally, if \((x^2 - x)^2 = 2 - (x^2 - x)\), then \(x^4 - 2x^3 + 2x^2 - x + 2 = 0\) and together with \(x^2 = -x\) we derive once again that \(x^2 = 2 + x\). Thus \(x^2 = -x\) is really superfluous, as claimed. For our security, the studied four cases, namely \(x^4 - 2x^3 + x = x^4 - 2x^3 + 2x^2 - x = 0\), \(x^4 - 2x^3 + x + z = 2 = 0\) and \(x^4 - 2x^3 + 2x^2 - x + 2 = 0\) in combination with \(x^2 = 2 + x\) will lead us to \(x^2 = x\) for the first two situations, or to \(x^2 = 2 + x\) for the other two ones, as required. So, \(x^2 = 2 - x\) is also redundant in \(R_1\), indeed. Secondly, we consider \(R_2\): Similarly to case (2) explored above, we can process in the same way by considering the element \(y + 2\) in order to get our initial claim that the equalities
$x^2 = -x$ and $x^2 = 2 - x$ are both redundant in $R_2$ as well. Thirdly, we consider $R_3$: Analogously to case (2) examined above, we can process in the same way by considering the element $y + 2$ getting that $R_3$ is isomorphic to $\mathbb{Z}_5$. However, in the ring $R_3$ the validity of the equality $x^2 = 2 - x$ is necessary and cannot be ignored. This ends our proof after all.

We finish off the work with some comments and a problem of interest. Foremost, one observes that the chief result Theorem 2.11 accomplished with Theorem 2.7 completely characterize those rings whose elements satisfy (one of) the equations $x^2 = x$ or $x^2 = -x$ or $x^2 = 2 + x$ or $x^2 = 2 - x$.

In view of Proposition 2.1, it may be relevant to wonder about the following:

**Problem 2.12.** Determine the structure of a ring whose elements satisfy the equations $x^5 = x$ or $x^5 = -x$.

Note that it is trivially seen that the field $\mathbb{Z}_5$ satisfies the above two equalities as well that they imply that $x^9 = x$. These rings are known to be commutative.

For rings in which $x^3 = x$ or $x^3 = -x$, called *weakly tripotent rings*, the interested reader can consult with [3], where a complete isomorphic characterization is given, whereas those rings in which $x^4 = x$ or $x^4 = -x$, called *weakly quadratent rings*, are completely described up to an isomorphism in [4].

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**References**


